THE

MATHEMATICAL GAZETTE.

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A CHAPTER ON ALGEBRA.

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VII.

The Exponential Expansion for Commensurable Indices.

If n is positive and commensurable, and $x^{\frac{1}{J}} = 1 + \frac{X}{J}$,

$$x^n \equiv \left(1+\frac{X}{J}\right)^I$$

$$\equiv \sum_0^I \frac{I(I-1)\dots(I-i+1)}{\lfloor i} \left(\frac{X}{J}\right)^i$$

$$\equiv \sum_0^I \frac{(nX)^i}{\lfloor i}. 1\left(1-\frac{1}{I}\right)\left(1-\frac{2}{I}\right)\dots\left(1-\frac{i-1}{I}\right).$$

Consider the coefficient of $\frac{(nX)^i}{\lfloor i}$.

(1) It is less than unity.

(2) It can be written $1\left(1-\frac{i-1}{I}\right).\left(1-\frac{1}{I}\right)\left(1-\frac{i-2}{I}\right)\dots.$

The sum of each pair of factors is the same, viz. $\left(2-\frac{i-1}{I}\right)$;

$\therefore$ by § I., the product of the outermost pair $1\left(1-\frac{i-1}{I}\right)$ is least;

$\therefore$ the coefficient is greater than $\left(1-\frac{i-1}{I}\right)^{\frac{i}{2}}$,

which, by § III., is greater than $1-\frac{i(i-1)}{2I}$.

Therefore, *if X is positive*,

$$x^n \text{ lies between } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ and } \sum_0^I \left\{1 - \frac{i(i-1)}{2I}\right\} \frac{n^i X^i}{\lfloor i}$$

This lower limit $$\equiv \sum_0^I \frac{n^i X^i}{\lfloor i} - \frac{n^2 X^2}{2I} \sum_0^{I-2} \frac{n^i X^i}{\lfloor i},$$

which $$> \left(1 - \frac{n^2 X^2}{2I}\right) \sum_0^I \frac{n^i X^i}{\lfloor i}.$$

$$\therefore\ x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ by less than } \frac{n^2 X^2}{2I} \sum_0^I;$$

but, since $$x^n > \left(1 - \frac{n^2 X^2}{2I}\right) \sum_0^I, \quad \therefore\ \sum_0^I < \frac{x^n}{1 - \frac{n^2 X^2}{2I}};$$

$$\therefore\ x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ by less than } \frac{\frac{n^2 X^2}{2I}}{1 - \frac{n^2 X^2}{2I}} \cdot x^n.$$

Now let I and J increase indefinitely, their ratio remaining n.

Then X takes the finite value $\log x$ and the above error becomes continuously smaller and ultimately vanishes.

Therefore, *when X is positive*, *i.e.* when $x > 1$,

$$x^n \equiv \sum_0 \frac{(n \log x)^i}{\lfloor i}.$$

If X is negative, *i.e.* if $x < 1$, the terms of

$$\Sigma \frac{(nX)^i}{\lfloor i} \ 1\left(1 - \frac{1}{I}\right) \ldots \left(1 - \frac{i-1}{I}\right)$$

are alternately $+$ and $-$; but it is still true that the series differs from

$$\Sigma \frac{n^i X^i}{\lfloor i} \text{ by less than } \Sigma \frac{i(i-1)}{2I} \frac{(-nX)^i}{\lfloor i}$$

where all the terms are positive;

$$\therefore\ x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ by less than } \frac{n^2 X^2}{2I} \sum_0^I \frac{n^i (-X)^i}{\lfloor i}$$

Now let $$x'^{\frac{1}{J}} \equiv 1 + \frac{-X}{J};$$

then by the former work,

$$\sum_0^I \frac{n^i (-X)^i}{\lfloor i} < \frac{x'^n}{1 - \frac{n^2 X^2}{2I}}.$$

But $x^{\frac{1}{J}} x'^{\frac{1}{J}} \equiv 1 - \frac{X^2}{J^2}$ which < 1;

$$\therefore\ x' < \frac{1}{x};$$

$$\therefore \text{ the error} < \frac{\frac{n^2 X^2}{2I}}{1 - \frac{n^2 X^2}{2I}} \frac{1}{x^n}.$$

Hence, for all positive commensurable values of n, whether I, J are finite or not,

$$x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i}, \text{ where } x^{\frac{1}{J}} \equiv 1 + \frac{X}{J},$$

by a quantity less than $\dfrac{\dfrac{n^2X^2}{2I}}{1-\dfrac{n^2X^2}{2I}} x^{\pm n}$,

the upper sign being used if $x > 1$ and the lower if $x < 1$.

If J is increased indefinitely, this error vanishes, and, for all positive values of x,

$$* x^n = \Sigma \frac{(n \log x)^i}{\lfloor i} \text{ when } n \text{ is positive.}$$

The case of a negative index presents no difficulty;

for $$x^{-n} = \left(\frac{1}{x}\right)^n = \Sigma \frac{\left(n \log \frac{1}{x}\right)^i}{\lfloor i} \text{ which} = \Sigma \frac{(-n \log x)^i}{\lfloor i}.$$

VIII.

The Binomial Expansion.

Let B_n stand for $x^n - \sum_0^\kappa (n)_i (x-1)^i$, where n has any value, and $(n)_i$ represents

$$\frac{n(n-1)\dots(n-i+1)}{1.2\dots i}.$$

The terms $(x-1)^i$ can be expanded in powers of x:

Assume that we so get

$$B_n \equiv x^n - [N_0 + N_1 x + N_2 x^2 \dots + N_\kappa x^\kappa],$$

where N_0, N_1... are algebraic functions of n of dimensions not greater than κ.

(1) If n is a positive integer less than κ (ν say), $B_n = 0$;

$$\therefore \; N_0, N_1 \dots \text{ all vanish when } n = \nu, \text{ except } N_\nu \text{ which} = 1.$$

Hence $$N_\nu = \frac{n(n-1)\dots(n-\kappa)}{n-\nu} . c_\nu,$$

where c_ν contains no n because the dimensions of the other factor are κ;

To find c_ν we have $N_\nu = 1$ when $n = \nu$;

$$\therefore \; c_\nu = \frac{1}{\{\nu . \nu - 1 \dots 1\} . \{(-)^{\kappa-\nu} . 1 . 2 \dots \kappa - \nu\}} \text{ or } \frac{(-)^{\kappa-\nu}}{\lfloor \kappa} (\kappa)_\nu.$$

(2) Again, B_n vanishes for all values of n if $x = 1$;

$$\therefore \; 0 = 1 - [N_0 + N_1 + \dots + N_\kappa].$$

Hence $$B_n \equiv N_0(x^n - 1) + N_1(x^n - x) \dots + N_\kappa(x^n - x^\kappa),$$

which by the above may be written $n(n-1)\dots(n-\kappa) \sum_{i=0}^{i=\kappa} c_i \dfrac{x^n - x^i}{n-i}$.

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* This expansion might have been more simply established by first confining ourselves to the case $n = 1$ (*i.e.* $I = J$ in the above work), and afterwards using the relation $\log x^n = n \log x$. The less simple proof has been given for the sake of the result when I and J are not indefinitely increased.

(3) To obtain limits for the value of $\sum_{i=0}^{i=\kappa} c_i \frac{x^n - x^i}{n-i}$, we use the Exponential Expansion established in § VII.

We have $$x^n \equiv 1 + \frac{nX}{\lfloor 1} + \frac{n^2X^2}{\lfloor 2} + \dots$$

$$\equiv 1 + nX\left[1 + \tfrac{1}{2} \cdot \frac{nX}{\lfloor 1} + \tfrac{1}{3} \cdot \frac{n^2X^2}{\lfloor 2} + \dots\right].$$

Now by § V., $$\frac{1}{i} = \underset{\theta=0}{\overset{\theta=1}{\text{Av.}}} |\theta^{i-1}|;$$

$$\therefore\ x^n \equiv 1 + nX \cdot \overset{1}{\underset{0}{\text{Av.}}}\left[1 + \theta nX + \theta^2 \frac{n^2X^2}{\lfloor 2} + \dots\right]$$

$$\equiv 1 + nX \overset{1}{\underset{0}{\text{Av.}}}(x^{n\theta});$$

$$\therefore\ \frac{x^n - x^i}{n-i} \equiv x^n \cdot \frac{x^{i-n} - 1}{i-n} \equiv x^n \cdot X \cdot \overset{1}{\underset{0}{\text{Av.}}} \left| x^{(i-n)\theta} \right|;$$

$$\therefore\ B_n = \frac{n \dots (n-\kappa)}{\lfloor \kappa} \cdot x^n X \overset{1}{\underset{0}{\text{Av.}}} \left| x^{-n\theta} \sum_{0}^{\kappa} (-)^{\kappa-i} (\kappa)_i x^{i\theta} \right|$$

$$\equiv n \cdot (n-1)_\kappa X \overset{1}{\underset{0}{\text{Av.}}} \left| x^{n\theta'} (x^\theta - 1)^\kappa \right|, \text{ where } \theta' \equiv 1 - \theta.$$

Now $$(x^\theta - 1)^\kappa < \overset{(x-1)^\kappa}{0};$$

$$\therefore\ B_n < n(n-1)_\kappa \left| \overset{(x-1)^\kappa}{0} \right| \cdot X \overset{1}{\underset{0}{\text{Av.}}} x^{n\theta},$$

and $$X \overset{1}{\underset{0}{\text{Av.}}} x^{n\theta} \equiv \frac{x^n - 1}{n}, \text{ as before.}$$

Hence $$x^n \equiv \sum_{0}^{\kappa} (n)_i (x-1)^i + \text{an error } (n-1)_\kappa (x-1)^\kappa \left| \overset{x^n - 1}{0} \right|.$$

This error vanishes as κ increases if $(x-1) < 1$ numerically.

In the special case of $x - 1 = 1$ or $x = 2$, the error vanishes ultimately if $(n-1)_\kappa$ vanishes.

Now the part of $(n-1)_\kappa$ after the signs of the factors have all become negative varies as $\left(1 - \frac{n}{\nu}\right)\left(1 - \frac{n}{\nu+1}\right) \dots \left(1 - \frac{n}{\kappa}\right)$, which, by § III., if $n > 0$

$$< \left(1 + \frac{1}{\nu}\right)^{-n} \left(1 + \frac{1}{\nu+1}\right)^{-n} \dots \left(1 + \frac{1}{\kappa}\right)^{-n} < \left(\frac{\nu}{\kappa+1}\right)^n;$$

and if $n = -f$, where $f < 1$,

$$\left(1 + \frac{f}{\nu}\right) \dots \left(1 + \frac{f}{\kappa}\right) > \left(1 + \frac{1}{\nu}\right)^f \dots \left(1 + \frac{1}{\kappa}\right)^f > \left(\frac{\kappa+1}{\nu}\right)^f;$$

$\therefore$ when $n > 0$ the error vanishes ultimately, but when $n < 0$ it increases indefinitely.

For the special case of $x - 1 = -1$ or $x = 0$ we must return to the beginning of the article.

We need only consider the case of $n > 0$, for otherwise x^n is not a finite quantity.

Thus, in this case, $B_n \equiv -N_0 = \frac{(n-1) \dots (n-\kappa)}{\lfloor \kappa} (-)^{\kappa+1}$, since $(\kappa)_0 = 1$.

This vanishes ultimately, as before, if $n > 0$.

IX.

On Incommensurable Indices.

The condition of obedience to the law $a^m \times a^n = a^{m+n}$ establishes the meaning of a^n when n is any commensurable, but leaves it undefined when n is incommensurable. Indicating the incommensurable by a Greek letter, we now seek to give a meaning to x^ν. Our freedom in giving such a meaning is limited only by the condition of continuity, viz. that if ν lies between two commensurables n and n', a^ν must be so defined that it lies between a^n and $a^{n'}$, whose meaning is already fixed.

Now we may suppose n to be $\frac{I}{J}$ and n' to be $\frac{I+1}{J}$.

Then as in § IV., $x^n = \left(1+\frac{nX}{I}\right)^I$ and $x^{n'} = \left(1+\frac{n'X}{I+1}\right)^{I+1}$,

where X is given by $x^{\frac{1}{J}} = 1+\frac{X}{J}$.

Now consider the expression $\left(1+\frac{\nu X}{I}\right)^I$.

(1) If X is positive,

since $\nu > n$, this $> \left(1+\frac{nX}{I}\right)^I$ and $\gtrless \left(1+\frac{n'X}{I+1}\right)^{I+1}$

according as $\left(1+\frac{\nu X}{I}\right)^{\frac{I}{I+1}} \gtrless 1+\frac{n'X}{I+1}$.

But by § III.,

$$\left(1+\frac{\nu X}{I}\right)^{\frac{I}{I+1}} < 1+\frac{\nu X}{I+1} \text{ and therefore} < 1+\frac{n'X}{I+1} \text{ since } \nu < n'.$$

Therefore when $x > 1$, $\left(1+\frac{\nu X}{I}\right)^I$ is a sound definition of x^ν, provided I and J are considered indefinitely great, and therefore X is $\log x$.

(2) If X is negative,

$$\left(1+\frac{\nu X}{I}\right)^I < \left(1+\frac{nX}{I}\right)^I$$

and $\left(1+\frac{\nu X}{I}\right)^I > \left(1+\frac{n'X}{I+1}\right)^{I+1}$, if $1+\frac{\nu X}{I} > \left(1+\frac{n'X}{I+1}\right)\left(1+\frac{n'X}{I+1}\right)^{\frac{1}{I}}$;

now $$\left(1+\frac{n'X}{I+1}\right)^{\frac{1}{I}} < 1+\frac{n'X}{I(I+1)};$$

$\therefore$ the last condition is satisfied if $1+\frac{\nu X}{I} > \left(1+\frac{n'X}{I+1}\right)\left(1+\frac{n'X}{I(I+1)}\right)$.

Remembering that X is negative,

this becomes $\nu < n' + \frac{n'^2 X}{(I+1)^2}$, *i.e.* $< n' + \frac{X}{J^2}$,

which is true if J is large enough.

Hence in this case also, the definition $x^\nu \equiv \left(1+\frac{\nu X}{I}\right)^I$, where I and J are taken indefinitely great, is valid.

Hence, reasoning as in § VII., we may say that

$$x^n \equiv 1 + n\log x + \frac{(n\log x)^2}{\lfloor 2} + \ldots.$$

for all values of n, commensurable or not.

It is easy to show that x^ν as so defined, satisfies the laws of indices.

Thus $x^\mu . x^\nu \equiv \left(1+\frac{\mu X}{I}\right)^I \left(1+\frac{\nu X}{I}\right)^I \equiv \left(1+\frac{\overline{\mu+\nu} . X}{I}\right)^I \left[1+\frac{\mu\nu X^2}{I^2+\mu+\nu . X_I}\right]^I$

$$= x^{\mu+\nu}\left[1+\frac{a}{I^2+bI}\right]^I \text{ say};$$

and, by § III., $\left(1+\frac{a}{I^2+bI}\right)^I > 1+\frac{a}{I+b}$ and $< \frac{1}{1-\frac{a}{I+b}}$;

and $\therefore$ (I being indefinitely great)$=1$.

Again, $\log x^\mu = \text{Lt}\, I\left[(x^\mu)^{\frac{1}{I}}-1\right] = \text{Lt}\, I\left[\left(1+\frac{\mu X}{I}\right)-1\right] = \text{Lt}\, \mu X = \mu \log x$;

$$\therefore\ (x^\mu)^\nu \equiv \left(1+\frac{\nu \log x^\mu}{I}\right)^I \equiv \left(1+\frac{\mu\nu \log x}{I}\right)^I \equiv x^{\mu\nu} \quad \text{by definition.}$$

Thus in the Exponential Expansion the number e can now be introduced: and we can prove $x = e^{\log x}$.

X.

On the Same Theorems for a Vector Variable.

The work in § VII. is applicable to vector values of x. The logarithm being defined exactly as before, takes the form $(\log a + \sqrt{-1}\,.\,\alpha)$, where a, α are the length and inclination of x. In the expression for the error, X and x are replaced by their moduli, and the error vanishes ultimately. The Binomial Theorem follows by § VIII. (provided that x is within two right angles of the line denoted by 1), the words 'lies between' at the end assuming a suitable meaning for vectors. The logarithmic expansion can be deduced from the exponential by expanding n^i in the finite algebraic form $\sum_{j=1}^{j=i} \left|(n)_j . i_j\right|$, whence $(x-1)^j \equiv \sum_{i=j}^{i=\kappa} \left|\frac{X^i}{\lfloor i} . i_j\right|$ with an error vanishing ultimately as κ is increased. Certain simple properties of the coefficient i_j establish the logarithmic expansion and other less simple expansions for higher powers of $\log x$.

HARROW, *July*, 1903. W. N. ROSEVEARE.

THE SLIDE RULE AND ITS USE IN TEACHING LOGARITHMS.

1. I should like at the outset to disclaim any intention of laying down the law dogmatically. The opinions expressed are stated definitely, but they are submitted with deference for your consideration.

My object is to argue two points:

(1) That the construction of a simple form of slide rule furnishes beginners with a good mode of approaching the subject of logarithms.

(2) That the use of the slide rule at any earlier stage than has been customary deserves every encouragement.

2. It is a commonplace remark that a principle may be presented to a pupil in such an abstract form that his mind fails to assimilate it. This

maxim is, however, not always respected. Galileo, after years of experimenting, reached clear notions about acceleration. There were great men before Galileo who never attained clearness on this point. Yet our textbooks polish off acceleration in a few paragraphs.

Even the Heuristic method, though it is in substance a protest against this form of error, is not, at any rate as presented by some of its advocates, entirely free from it. For example, a boy is invited, after contemplating a lump of coal for five minutes, to formulate a statement. The suggestion is that he will be moved to declare that "matter cannot set itself in motion." That is, the ordinary boy is to arrive in five minutes at a statement, the precise formulation of which is an item in the list of Newton's titles to fame.

These authorities—with sanguine ideas as to the ability of the average pupil—not unnaturally adopt as their cardinal rule "never teach anyone anything," an odd principle for a teacher. The better maxim is never teach a pupil that which he can find out for himself with a reasonable expenditure of time, patience and pains.

This is the true rule, because the object of education is to enable people to find out things for themselves. The application of this guiding rule is not easy; it is apt to be misunderstood and resented by pupils who have been brought up on the plan of being shewn everything; it does not pay in examinations, it results in less ground being covered; and it requires an amount of time, effort, and individual attention from the teacher which it is often, under existing conditions, simply impossible for him to give, but we all now see that it should be our ideal.

But if the plan of making the pupil cut some of the steps for himself is to be followed, the ascent, for the great majority of pupils, must be a very easy one and must consequently appear somewhat long. In the end, however, these slow experimental methods will prove to surpass purely didactic teaching in educating as distinguished from information-inserting.

3. I suggest that it is possible to approach the thorny subject of logarithms in an experimental fashion with advantage. I call the subject of logarithms a thorny one because it is within my personal knowledge that to many successful Woolwich candidates calculation by logarithms is an occult process, savouring of the black art.

How many times I have been asked how it is that

$$\log(a+b) \text{ is not equal to } \log a + \log b,$$

and wondered whether a zealous attempt to explain has had any effect when the many previous zealous and skilful explanations which candidates received at school have obviously had none. This fact suggests that the subject might perhaps be approached with advantage in a different manner.

For instance, let us begin by trying to construct a machine to do addition and subtraction. After a discussion, in which the teacher will have to be the predominant partner, we arrive at something like this: each boy using a pair of paper slips made by himself, in which a step of, say, an inch along the uniformly divided scale brings us to a number which is 2 more than the one we started with.

			0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10			
A			*P*					*B*					

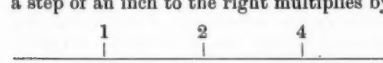
If the strips* are shown overlapping as above we see that they mechanically

*A couple of wooden slips with figures stencilled on them were used for this and subsequent illustrations. Better than stencils are the indiarubber figures and letters sold for printing shop window tickets. I am indebted to Mr. J. H. Naylor, R.M. Academy, for this 'tip,' most useful when diagrams are often wanted.

add 3 to 5; for instance, for $AP+PB=AB$. The same setting mechanically subtracts 5 from 8.

Stated in a slightly different way. . . . For any 'setting' or relative position of the strips the DIFFERENCE between any corresponding pair of numbers on the scales is constant.

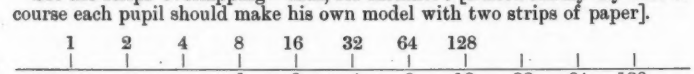
4. We next suggest the problem of making a machine for multiplication, and arrive at the idea that, as in the addition machine, a step of say an inch to the right *adds* 2 we will try whether we cannot make a machine in which a step of an inch to the right multiplies by 2.



This leads to a uniformly divided scale graduated as shewn.

We are now able to work out, using a pair of such scales, certain arithmetical problems—in fact we have a rudimentary slide rule.

Set the strips overlapping—thus, for instance: [I need hardly say that of course each pupil should make his own model with two strips of paper].



Notice that any number on the upper strip is 8 times that just below it, and that hence we can, from this particular 'setting' or relative position of the strips, read off the answers to certain sums in arithmetic thus:

MULTIPLICATION.—The fact that 8 times 4 is 32 is shown thus: $\begin{matrix} 8 & 32 \\ 1 & 4 \end{matrix}$.

DIVISION.—The fact that 32 divided by 4 is 8 is shown in the same way.

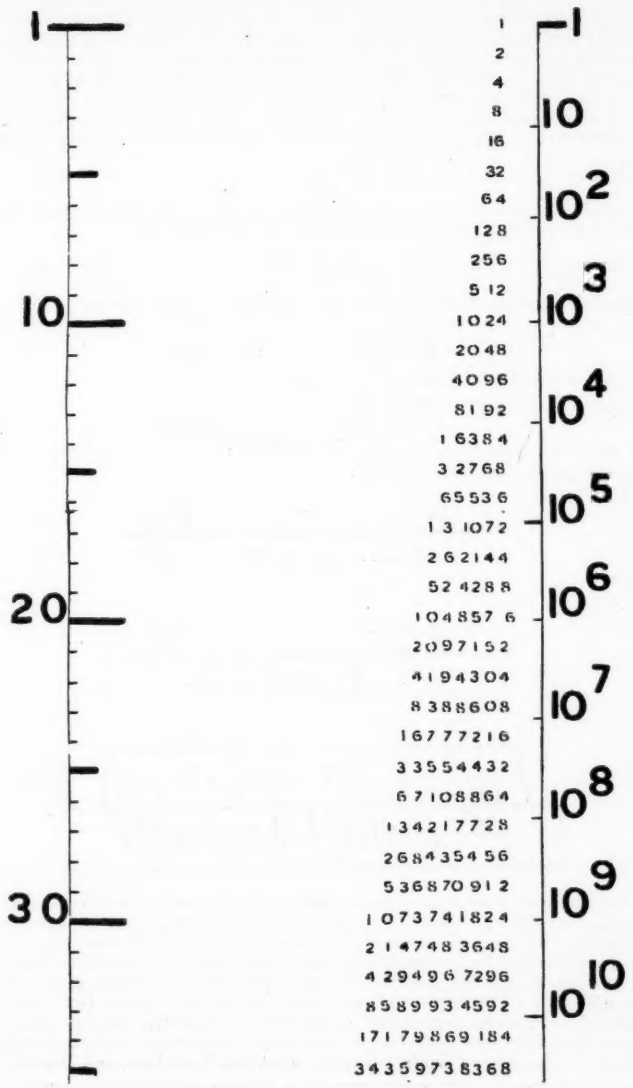
PROPORTION.—The fact that 32 is to 4 as 128 is to 16 is shown thus: $\begin{matrix} 32 & 128 \\ 4 & 16 \end{matrix}$. The scales may, of course, be extended to the left if desired.

Other sums may now be tried in a similar way, but with the strips of paper set to a different overlap. We observe that in this case—For any 'setting' or relative position of the strips the RATIO of any corresponding numbers* on the scales is constant.

It is evident that in its present form the instrument is not a practical one, for only a few numbers can be dealt with. We could clearly construct another instrument to deal with the numbers 3, 9, 27, 81, etc., but this does not overcome the difficulty. We made our scales on the supposition that a step of an inch to the right effected multiplication by 2. It is clear that a step of 2 inches to the right multiplies by 4, of 3 inches to the right multiplies by 8, and so on, and we infer that any step of definite amount corresponds to multiplication by some definite number. The problem then is reduced to finding the proper position for any required number on the scales.

We have now got to the pinch of the case. A little time given to the class is certain to produce some good and some poor suggestions. For example, you are certain to be told that 3 should come half way between 2 and 4. The line to follow will depend on the suggestions received—so the following is only one of many possible modes of advance. Let us consider diagram (i) supposed, to begin with, to contain the marks for 10, 100, 1000, 10000 and so on. These marks will, as we have seen, be all equidistant, say a decimetre apart. Now, looking at the diagram which is at present a

* Note that if we invert one of the adding scales we get the SUM of corresponding numbers constant, and if we invert one of the multiplying scales we get the PRODUCT of corresponding numbers constant.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65536
131072
262144
524288
1048576
2097152
4194304
8388608
16777216
33554432
67108864
134217728
268435456
536870912
1073741824
2147483648
4294967296
8589934592
17179869184
34359738368
1
10
10^2
10^3
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10^5
10^6
10^7
10^8
10^9
10^{10}

multiplying machine with 10, 100, and so on marked on it, we want to know where to put the mark for 2. Suppose the distance from 1 to 2 ought to be x decimetres, we have to find x. Evidently the distances 2-4, 4-8, 8-16, . . . should each be equal to x. Now if we take any value for x and mark in the series of numbers as shewn in the diagram, it is clear that all the numbers shewn, both the multiples of 10 and the multiples of 2 should always come in order. A larger number should never be reached before a smaller one. Hence we must reach 1000, the mark for which is 3 decimetres from A, before we reach 1024, the mark for which should be $10x$ decimetres from A.

Hence $$x > \frac{3}{10} > \cdot 3.$$

Again we must reach the point which should be $33x$ decimetres from A before we reach the point which is 10 decimetres from A.

Hence $$x < \frac{10}{33} < \cdot 303.$$

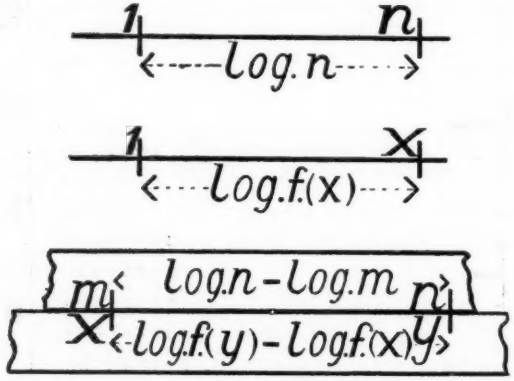
Inviting attempts to find x still more closely, we soon find that

$$103x > 31, \qquad x > \cdot 30097,$$
$$93x < 28, \qquad x < \cdot 30107\ ;$$

or, if it is considered worth while* to go further,

$$196x > 59, \qquad x > \cdot 3010204,$$
$$485x < 146, \qquad x < \cdot 3010309.$$

Of course all this is nothing but $2^{103} > 10^{31}$, and so on, which, however, is in too abstract a form for beginners.



We may now give x its technical name, logarithm of 2, and the calculation of the four figure logarithms of 3, 4, 5, 6, 7, 8, 9, might be undertaken—affording a lesson in the art of saving labour by thinking, for only log 3 and log 7 require any fresh calculations—and even these can be found from the diagram by a little ingenuity. It is my firm conviction that a great part of the difficulty which boys find in logarithms is due to the fact that the mode of calculating logarithms is to them a mystery. If they are once shewn how any logarithm *can* be calculated, they will, I think, make no difficulty in accepting the fact that the work has been done, once for all, by

* Shanks's table of powers of 2 is fairly accessible.

others. We are now in a position to construct a primitive slide rule and to work out a number of questions on it.

5. I now come to the utility of the slide rule. A distinguished French mathematician has spoken of "that inexplicable prejudice against the slide rule, against which it is impossible to protest too strongly." This prejudice is dying out, but it exists. The Diagram (2) shews the mathematical principle of the ordinary slide rule so far as it goes beyond what we have already discussed. You will observe that if the scales 1 and 2 are placed in conjunction with their units coinciding we must have $f(x)=n$, so that two such scales furnish us with a table of 'natural' values of $f(x)$. The diagram (2) also shews that we have the means of multiplying or dividing by $f(x)$, in fact of working the equation

$$\log n - \log m = \log f(y) - \log f(x), \text{ or } \frac{m}{f(x)} = \frac{n}{f(y)}.$$

It is clear that with such an instrument we can work out any question which can be done with logarithms; and several which can not. The slide rules in the market usually contain two scales for $\log n$, one on the ruler and one on the slide and logarithmic scales for some of the following functions:

$f(x)=x^2$, x^{-1}, $\sin x$ (which of course serves also for $\cos x$), $\tan x$, $\log x$, and a plain scale.

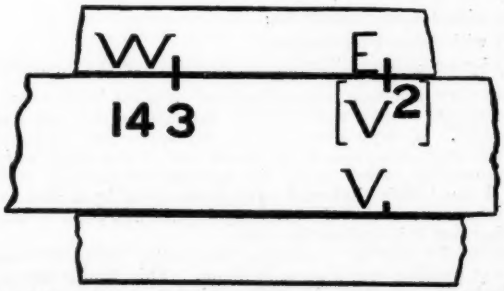
The $\frac{1}{x}$ scale is not very much needed, for we obtain it at once by putting the ordinary slide upside down.

Now, I claim for the use of the slide rule many advantages.

First. In very many cases an isolated numerical example does not bring out the essence of a formula. We want to observe the effect on the result of changes in the data. With a slide rule we can work out fifty numerical examples quicker than the answers can be written down.

The K.E. of a shell weighing W lbs. moving at V f.s. is

$$\frac{WV^2}{2g \times 2240} \text{ ft. tons} = \frac{WV^2}{143000} \text{ ft. tons.}$$



The diagram (3) shows how, taking an initial velocity of 2700 f.s. and a given weight of shell, we can read off the remaining K.E. E at any remaining velocity V. People who do not think say that the slide rule in these cases prevents a boy from thinking. We must distinguish. When I am trying to teach a boy about energy I don't want him to be thinking of multiplication and division; I welcome any means of preventing him from thinking about multiplication and division. I admit that the slide rule economises useless thought.* I consider this a merit.

* Mach., *Science of Mechanics*, p. 488, urges this point very forcibly.

6. Second, the slide rule familiarises us with the values of the trigonometrical functions, and enables *really accurate* graphs to be plotted very quickly. For instance, diagrams of Fourier series* can be rapidly and accurately drawn showing each term and the result of compounding any number of terms.

I have, we all have, again and again been told that there is no angle whose tangent is 3, no angle whose sine is $\frac{1}{4}$.

Mr. Siddons spoke in January of the undue importance which has been attached to 18°, 30°, 45° as compared with other angles equally worthy of respect. But that is not the worst: a gentleman has actually printed the formula, for boys to learn,

$$\sin 20\tfrac{1}{4}^\circ = \frac{\sqrt{3}}{5} \text{ nearly.}$$

We have all seen a good many queer formulæ, but I confess I have never seen anything to equal this. The use of a slide rule renders such nonsense impossible.

The third advantage I claim is certainty. In real life you generally needn't do a question by the shortest way, nor very quickly; but it must be right. With a slide rule there is no excuse for not checking your work.

Cube roots, plane and spherical triangles, quadratic and cubic equations, can with a little management be worked out independently with a slide rule, but at any rate they can be very easily checked by it.

Fourthly, how many authorities have denounced the practice of students who assert that the circumference of a penny bun is

$$9{\cdot}734933\dot{7}62749 \text{ inches,}$$

or that the angle of inclination of a shaky chimney-pot is 84° 17′ 34·2″. Yet how invariably boys make this mistake,† and how difficult it is to persuade them that it is a mistake. They call it accuracy.

If a class work out the value of $\sqrt{m \cdot n}$,

(i) graphically on a small and large scale;
(ii) with a slide rule;
(iii) with 4 figure logarithms;
(iv) with 7 figure logarithms;
(v) by actual multiplication and extraction of the square root;

they cannot very well help seeing that each method has its own degree of accuracy, and that to give results which the method does not warrant is to tell an untruth. By supposing m and n only approximately known, we can see when the inevitable error in the result due to the imperfections of the data is appreciably added to by the approximate method of calculation adopted. I can testify that such work undertaken by a class arouses considerable interest; and that the necessary logarithmic work is done without much feeling that it is against the collar.

It is noteworthy that the exact error (due to the slide rule) committed in working any problem can always be found. This familiarises pupils with some of the leading results in the mathematical theory of errors; for instance, as has been already implied—that, to do justice to data, the Probable Error due to the use of approximate methods of calculation should not exceed about $\frac{1}{4}$ of the P.E. due to the imperfections of the data.

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*Such diagrams are given in Byerley, "Fourier Series," p. 63; or Merriman & Woodward, *Higher Mathematics* (article by Professor Byerley), p. 199.

†Of course "It's near enough" is a conclusion pupils need no encouragement to adopt. But do not let us teach accuracy by inaccuracy. There are occasions when 7 or even 10 figure logarithms are called for; geodesic surveys furnish plenty of instances; but do not let us bring in 7 figure logarithms 'nisi dignus vindice nodus inciderit.'

Again, let us find the cube roots of the numbers 2, 3, 4, 5, 6, 7, 9...with a slide rule. Here we have an exact graduation to set the slide to; and we find, for instance, that the P.E. of the result is, say, a unit in the fourth significant figure. Now let us take the cube roots of 2·16, 3·16, 4·16.... An additional source of error arises from the fact that we have to estimate, or interpolate, for the position of the 6. This causes the P.E. of the result to be perceptibly larger. Such examples, which may be multiplied almost indefinitely, give a real insight into the way in which a result is affected by several component sources of error.

There is perhaps no paramount scientific principle so often disregarded as this—always have some idea of the degree of accuracy of which your method is capable, and of the degree of accuracy of your own work.

To take a single case—our whole national system of competitive examination rests on a total disregard of this principle. This, however, is bordering upon topics of controversy.

I have one thing to say in conclusion. If anything in this paper is of value, the credit is largely due to Colonel H. C. Dunlop, R.A., with whom I have constantly discussed slide rule problems for some years.

The Logocyclic Slide Rule.—The special object of the logologarithmic slide is to enable the user to read off powers of numbers in the same way as the common slide rule enables us to read products.

Putting $f(x)=\log x$ in the principal equation, we have at once from the diagram $\log\log y-\log\log x=\log n-\log m$ or $y^m=x^n$, while, as special cases, if $m=1$, $y=x^n$, and if $x=10$, $n=\log_{10}y$. Thus powers and logarithms can be read off at a glance.

Numerous applications at once suggest themselves. As an instance of the use of the slide in getting out empirical formulæ, we may take Dr. Vincent's formula (which, perhaps, ought hardly to be termed empirical), viz.: atomic weight of the n^{th} element, when the elements are arranged in order of their atomic weights, is (in general) $(n+2)^{1\cdot21}$.

It is a singular fact that the logolog scale has repeatedly been invented, forgotten, and reinvented. The first inventor was Dr. Roget. It was rediscovered and patented by Captain Thomson, R.A.; again, in Germany, by M. Blanc (Dyck's *Catalog*, p. 145), and again, almost simultaneously, reinvented by Professor Perry, and reintroduced by Colonel Dunlop and myself. The use of a $\log(-\log x)$ scale for numbers less than unity was one fondly thought new, but in this idea, which Professor Perry also brought forward, we were all anticipated by M. Blanc.

R. M. ACADEMY, WOOLWICH. C. S. JACKSON.

MATHEMATICAL NOTES.

128. [R. 6. b.] *On Gauss' Principle of Least Constraint and the Equations of Mechanics.*

The object of this note is to point out the remarkably simple principle to which Gauss' principle leads. The new principle is much simpler—not being an integral principle—than that of Hamilton,

$$\delta\int_{t_0}^{t_1}(T+U)dt=0,$$

but, like it, is independent of coordinates, and leads at once to equations in generalised coordinates which have important properties.

1. Consider any material particle of mass m_ν, whose coordinates are (x_ν, y_ν, z_ν) at time t, and which forms part of a system. If m_ν were quite

free and (X_ν, Y_ν, Z_ν) were thus the only forces acting on it, its coordinates at time $t+dt$ would be

$$\left.\begin{aligned} &x_\nu + \dot{x}_\nu dt + \tfrac{1}{2}\frac{X_\nu}{m_\nu}dt^2, \\ &y_\nu + \dot{y}_\nu dt + \tfrac{1}{2}\frac{Y_\nu}{m_\nu}dt^2, \\ &z_\nu + \dot{z}_\nu dt + \tfrac{1}{2}\frac{Z_\nu}{m_\nu}dt^2. \end{aligned}\right\} \quad \ldots\ldots(1)$$

However, owing to the connexions, the actual coordinates at the time $t+dt$ are

$$x_\nu + \dot{x}_\nu dt + \tfrac{1}{2}\ddot{x}_\nu dt^2. \quad \ldots\ldots(2)$$

The square of the difference between the positions (1) and (2) is then

$$\frac{dt^4}{4}\left\{\left(\frac{X_\nu}{m_\nu} - \ddot{x}_\nu\right)^2 + \left(\frac{Y_\nu}{m_\nu} - \ddot{y}_\nu\right)^2 + \left(\frac{Z_\nu}{m_\nu} - \ddot{z}_\nu\right)^2\right\}.$$

Now, Gauss' principle* states that the sum of these squared deviations, each multiplied by the proper mass, is a minimum. That is to say, the actual accelerations make the sum a minimum; or, for our purpose, the variation, which must affect only the accelerations and not the positions or velocities, must be zero. Thus, if δ denote such a variation, Gauss' principle becomes

$$\delta \sum_\nu \{(X_\nu - m_\nu \ddot{x}_\nu)^2 + \ldots\} = \sum_\nu \{(X_\nu - m_\nu \ddot{x}_\nu)\,\delta\ddot{x}_\nu + \ldots\} = 0. \quad \ldots\ldots(3)$$

There is an analogy with the variational form of d'Alembert's principle, but it must be remembered that here the δ's are of different meaning. Here we have

$$\delta x_\nu = \delta y_\nu = \ldots = \delta\dot{x}_\nu = \delta\dot{y}_\nu = \ldots = 0,$$

but $\delta\ddot{x}_\nu, \ldots$ are not zero; while, with d'Alembert's variations, $\delta x_\nu, \ldots$ are not zero.

2. The equation (3) may be written

$$\begin{aligned} \sum_\nu (X_\nu \delta\ddot{x}_\nu + Y_\nu \delta\ddot{y}_\nu + Z_\nu \delta\ddot{z}_\nu) &= \sum_\nu m_\nu (\ddot{x}_\nu\, \delta\ddot{x}_\nu + \ddot{y}_\nu \delta\ddot{y}_\nu + \ddot{z}_\nu \delta\ddot{z}_\nu) \\ &= \tfrac{1}{2}\delta \sum_\nu m_\nu (\ddot{x}_\nu{}^2 + \ddot{y}_\nu{}^2 + \ddot{z}_\nu{}^2). \quad \ldots\ldots(4) \end{aligned}$$

If now we denote $\quad \tfrac{1}{2}\sum_\nu m_\nu (\ddot{x}_\nu{}^2 + \ddot{y}_\nu{}^2 + \ddot{z}_\nu{}^2),$

which is independent of coordinates, by $\ddot{\mathbb{T}}$, in analogy with the kinetic energy T; and assume, for simplicity, a force-function $\ddot{U}$ to exist, so that

$$\delta\ddot{U} = \sum_\nu (X_\nu \delta\ddot{x}_\nu + \ldots);$$

the equation (4) takes a form independent of coordinates,

$$\delta\ddot{U} = \delta\ddot{\mathbb{T}} \quad \text{or} \quad \delta(\ddot{\mathbb{T}} - \ddot{U}) = 0. \quad \ldots\ldots(5)$$

This form may now play the part of Hamilton's principle in the deduction of equations of motion in generalised coordinates, but the use of (5) is still simpler than the use of Hamilton's principle.

3. Let $q_1, q_2, \ldots, q_n$ be the generalised coordinates of the system; then $\ddot{U}$ is a function of $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \ddot{q}_1, \ldots, \ddot{q}_n$, and $\ddot{\mathbb{T}}$ of these and t, explicitly. Remembering that the δ affects only the accelerations, (5) gives

$$\sum_{\nu=1}^{n} \left\{\frac{\partial\ddot{\mathbb{T}}}{\partial\ddot{q}_\nu} - \frac{\partial\ddot{U}}{\partial\ddot{q}_\nu}\right\}\delta\ddot{q}_\nu = 0.$$

* "Ueber ein neues allgemeines Grundgesetz der Mechanik" (1829; *Werke*, v., 23-28).

This is equivalent to the system

$$\frac{\partial \mathfrak{T}}{\partial \ddot{q}_\nu} - \frac{\partial \ddot{U}}{\partial \dddot{q}_\nu} = 0. \quad (\nu = 1, 2, \dots n.)$$

We at once verify that $$\frac{\partial \ddot{U}}{\partial \dddot{q}_\nu} = \frac{\partial U}{\partial q_\nu};$$

so that the equations of motion of a system have the simple form

$$\frac{\partial U}{\partial q_\nu} = \frac{\partial \mathfrak{T}}{\partial \ddot{q}_\nu}. \quad (\nu = 1, 2, \dots n.) \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

4. Mr. Whittaker remarked to me that the equations (6) had been already given by Appell. In fact, in 1899 and 1900, Appell published* several notes drawing attention to the remarkable fact that the equations (6)—which he obtained in quite a different way from mine—apply to the cases of non-holonomous systems (*i.e.* in which the equations of condition are non-integrable differential equations; these occur, *e.g.* in cases of rolling motion), in which Lagrange's equations fail. I shall show this in another manner, which appears to me not uninstructive. In fact, I shall show that the equations (6) are equivalent to an extended form of Lagrange's equations†—a form which reduces to the usual form when the system is holonomous.

5. Suppose the equations of condition are given in the non-integrable form

$$dx_r = \sum_{\nu=1}^{n} Q_{r,\nu}\, dq_\nu + Q dt, \dots; \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

so that $$\dot{x}_r = \sum_{\nu=1}^{n} Q_{r,\nu}\, \dot{q}_\nu + Q, \dots; \text{ and } \frac{\partial \ddot{x}_r}{\partial \ddot{q}_\nu} = \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu}.$$

If the equations (7) are integrable, and we can then assert that

$$Q_{r,\nu} = \frac{\partial x_r}{\partial q_\nu},$$

we also have $$\frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} = \frac{\partial x_r}{\partial q_\nu}.$$

Then, putting $$w_r^2 = \ddot{x}_r^2 + \ddot{y}_r^2 + \ddot{z}_r^2,$$

we have
$$\frac{\partial \mathfrak{T}}{\partial \ddot{q}_\nu} = \tfrac{1}{2} \sum_r m_r \left(\frac{\partial w_r^2}{\partial \ddot{x}_r} \cdot \frac{\partial \ddot{x}_r}{\partial \ddot{q}_\nu} + \frac{\partial w_r^2}{\partial \ddot{y}_r} \frac{\partial \ddot{y}_r}{\partial \ddot{q}_\nu} + \frac{\partial w_r^2}{\partial \ddot{z}_r} \frac{\partial \ddot{z}_r}{\partial \ddot{q}_\nu} \right)$$
$$= \sum_r m_r \left(\ddot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} + \dots \right) = \sum_r m_r \left\{ \frac{d}{dt} \left(\dot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} + \dots \right) - \left(\dot{x}_r \frac{\partial \ddot{x}_r}{\partial \dot{q}_\nu} + \dots \right) \right\}$$
$$= \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \sum_r m_r \left(\dot{x}_r \frac{\partial \ddot{x}_r}{\partial \dot{q}_\nu} + \dots \right),$$

where $$T = \tfrac{1}{2} \sum_r m_r (\dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2).$$

Hence $$\frac{\partial \mathfrak{T}}{\partial \ddot{q}_\nu} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \sum_r m_r (\dot{x}_r \dot{Q}_{r,\nu} + \dot{y}_r \dot{Q}'_{r,\nu} + \dot{z}_r \dot{Q}''_{r,\nu});$$

and so Lagrange's equations, generalised for non-holonomous systems, are

$$\frac{\partial U}{\partial q_\nu} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \sum_r m_r (\dot{x}_r \dot{Q}_{r,\nu} + \dots). \qquad \dots\dots\dots\dots\dots\dots\dots\dots (8)$$

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* *Comptes Rendus*, t. cxxix. (1899), pp. 317-320, 459-460; *Crelle's Journal*, Bd. cxxi. (1900), pp. 310-319; Bd. cxxii. (1900), pp. 205-209.

† Ferrers: "Extension of Lagrange's Equations," *Quart. Journ.*, vol. xii. (1873), pp. 1-5.

Now if, and only if, the conditions (7) are total differentials,

$$Q_{r,\nu}=\frac{\partial x_r}{\partial q_\nu},\quad Q'_{r,\nu}=\frac{\partial y_r}{\partial q_\nu},\quad Q''_{r,\nu}=\frac{\partial z_r}{\partial q_\nu};$$

and the last term of (8) reduces to

$$\frac{\partial T}{\partial q_\nu},$$

and (8) takes the usual Lagrangian form.

The equations (6) are then equivalent to the system (8), and have the advantage of being expressed wholly in generalised coordinates; while Lagrange's equations,

$$\frac{\partial U}{\partial q_\nu}=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\nu}\right)-\frac{\partial T}{\partial q_\nu},\quad (\nu=1, 2, \ldots n)$$

result from (6) or (8) for the special case of holonomous systems.

PHILIP E. B. JOURDAIN, B.A., Trinity College, Cambridge.

129. [K. 20. e.] *Two Trigonometrical Notes.*

I. There is a large class of formulas by which one length (which may be supposed unknown) is expressed in terms of another length (which may be supposed known), and of the trigonometrical ratios of certain angles. Such formulas are often easily established by the aid of the artifice of expressing the ratio of the unknown to the known as a product of two or more ratios, each of which is either a trigonometrical ratio of a known angle or is the ratio between two sides of a triangle whose angles are known, and therefore expressible as the ratio between two known sines.

For example, take the problem: *To express the in-radius of a triangle in terms of one side and two angles.* With the usual notation we have to express r in terms, say, of a, B, C.

From I, the in-centre, let ID be drawn perpendicular to BC. We have

$$\frac{r}{a}=\frac{ID}{BC}=\frac{ID}{IC}\cdot\frac{IC}{BC}=\sin ICD\times\frac{\sin IBC}{\sin BIC}$$

$$=\sin\frac{C}{2}\times\frac{\sin\frac{B}{2}}{\sin\left(180^\circ-\frac{B}{2}-\frac{C}{2}\right)}=\frac{\sin\frac{C}{2}\sin\frac{B}{2}}{\sin\frac{B+C}{2}}.$$

Hence the well-known result,

$$r=a\sin\frac{B}{2}\sin\frac{C}{2}\Big/\cos\frac{A}{2}.$$

Again take this problem: *Given the distance between two points A, B on a horizontal plane, and the angles of elevation α, β of a point C in the vertical plane containing AB, as seen from A and from B respectively, to find the vertical height of C above the level of AB.*

Let CD be the perpendicular from C on AB produced.

Then
$$\frac{CD}{AB}=\frac{CD}{CA}\cdot\frac{CA}{AB}=\sin CAD\times\frac{\sin CBA}{\sin ACB}$$

$$=\sin\alpha\times\frac{\sin(180^\circ-\beta)}{\sin(\beta-\alpha)};$$

$$\therefore\ CD=AB\sin\alpha\sin\beta/\sin(\beta-\alpha).$$

It would be easy to multiply examples. These two form part of the usual bookwork of elementary trigonometry, but are often proved in the

text-books by longer methods, and without any clue by which the beginner could invent a proof for himself.

It will be seen that we have to choose one or more lengths in the figure as intermediaries between the known and the required lengths, in such a way that the ratio of unknown to known is expressible identically as compounded of known ratios.

II. A large proportion of the exercises usually set on heights and distances, not confined to one plane, are reducible to easy *plane* problems by projecting the figure on the horizontal plane, and expressing the horizontal lines in terms of some height or distance, given or required, and the trigonometrical ratios of given angles of elevation. An easy example will suffice to indicate the method.

From a station A, a balloon is observed at P at an elevation of $\theta°$, its bearing being N. The balloon is known to be moving horizontally at the rate of 6 miles per hour. After $\frac{1}{4}$ hour it is seen from A at a point Q in direction N.W. at an elevation of $\phi°$. Find h, the height of the balloon in miles.

Let B and C be the 'horizontal projections' of P and Q. Then

$$AB = h\cot\theta°,\quad AC = h\cot\phi°,\quad BC = PQ = 1\tfrac{1}{2} \text{ miles, and } B\hat{A}C = 45°.$$

In the triangle BAC we have to relate the elements AB, BC, CA, $B\hat{A}C$, which is done by the formula

$$BC^2 = AB^2 + AC^2 - 2AB\,.\,AC\cos BAC.$$

This gives $\quad \frac{9}{4} = h^2\cot^2\theta° + h^2\cot^2\phi° - h^2\cot\theta°\cot\phi°\sqrt{2}.$

Hence h in terms of the data. R. F. MUIRHEAD.

130. [A. 1. b.] *Note on Mr. Roseveare's "Chapter on Algebra."*

In Section III. inequalities equivalent to those in Chrystal's *Algebra*, Chapter XXIV., § 7, are deduced in an interesting way from the results of Section II.; but perhaps the proof given in 'Chrystal,' depending on the inequality $(p-q)\{q(x^p-1)-p(x^q-1)\}\not<0$ (x being positive), is more natural. The most troublesome part of Chrystal's procedure, namely, the proof of the last-mentioned equality, may be simplified by substituting the following proof, the idea of which was suggested to me some time ago by a friend.

If x is positive, and i any positive integer,

$$\begin{aligned}
&i(x^{i+1}-1)-(i+1)(x^i-1)\\
&\quad=(x-1)\{i(x^i+x^{i-1}+x^{i-2}+\ldots+x+1)-(i+1)(x^{i-1}+x^{i-2}+\ldots+x+1)\}\\
&\quad=(x-1)\{ix^i-x^{i-1}-x^{i-2}-\ldots-x-1\}\\
&\quad=(x-1)\{(x^i-x^{i-1})+(x^i-x^{i-2})+\ldots+(x^i-x)+(x^i-1)\}\\
&\quad=(x-1)^2\{x^{i-1}+x^{i-2}(x+1)+x^{i-3}(x^2+x+1)+\ldots\\
&\qquad\qquad+(x^{i-1}+x^{i-2}+\ldots+x+1)\}\\
&\quad=(x-1)^2\{ix^{i-1}+(i-1)x^{i-2}+(i-2)x^{i-3}+\ldots+2x+1\},
\end{aligned}$$

which is $\not<0$.

Thus $$\frac{x^{i+1}-1}{i+1}>\frac{x^i-1}{i}>\frac{x^{i-1}-1}{i-1}\ldots>\frac{x^j-1}{j}$$

if j is an integer <1.

Hence $$(i-j)\{j(x^i-1)-i(x^j-1)\}\not<0.$$

The extension to the cases in which i and j are replaced by any commensurable positive or negative fractions can then be carried out as in 'Chrystal.'

Remark.—The inequality has been demonstrated by expressing the excess of the greater over the less as the product of an essentially positive factor and the square of the difference between the two quantities (x and 1), which

being made equal will reduce the inequality to equality. Very many algebraic inequalities (perhaps all rational integral ones which become equalities for critical values of the quantities involved) can be demonstrated by expressing the difference in that form, or in a sum of expressions of that form. (See, for example, Roseveare's *Chapter*, § II.) R. F. MUIRHEAD.

REVIEWS.

A Mathematical Solution Book. By B. F. FINKEL. Pp. xvi., 549 (4th Edition.) (Kibler & Co., Springfield, Missouri, U.S.A.)

The care with which this little book has been written and its freedom from typographical errors have proved so acceptable to the youth of America that three editions of twelve hundred copies have now been exhausted,—and yet they want more. The author is the well-known Professor of Mathematics and Physics at Drury College, and his aim is to provide systematic solutions of difficult questions in the earlier subjects. He is very averse from "Short Cuts" and "Lightning Methods"; and insists that solutions should be written out step by step in logical order and the chain of reasoning made complete in every link. Possibly the logical instinct is more highly developed there than it is here, where the utter impossibility of comprehending the simplest reasoning is too often apparent in young pupils. "Do this, ye shall know why hereafter" is the only way [*v.* article in *Engineering* of June 19th, 1903, entitled "Reform in Mathematical Education," inspired by Professor Perry].

Of 500 pages, 200 are devoted to Arithmetic, 200 to Mensuration (which evidently from its utilitarian nature looms very large indeed), 50 to Geometry (to which injury a further insult is added by filching 10 for Logic) and 50 to Algebra interspersed with Biographies.

The weights and measures used throughout the book are for the most part uncompromisingly English; but questions on money are worked in U.S. currency and on specific gravities in the metric system. Our old friend, the rod of $5\frac{1}{2}$ yards (whom we would so gladly spare in England) comes up smiling; and so does Apothecaries' weight. The most appalling accuracy prevails through the book,—the area of a certain circle is 12·566368 sq. ft. and π is given to 600 places of decimals (p. 262). How much more useful would be Professor Lodge's approximations for this and other multipliers! The largest known primes and perfect numbers are given on p. 42 (curious results, tricks to show the stretch of human brain), but one looks in vain for a table of prime numbers or factors. On pp. 554-5 there are tables of various constants, square and cube roots, natural logarithms, degrees into circular measure, etc.

We are perfectly willing to accept Mr. Workman's ironical statement that if we assume N to be the number required we are keeping within the limits of Arithmetic, whereas if we assume x to be the number we immediately remove our solution into the sphere of Algebra. What then must we think of this solution (p. 147)? "Had an article cost 10% less, the number of % gain would have been 15% more; what was the gain? Assume 100% = selling price, 100% = cost price. Then $1\frac{1}{2} \times \{100\% - (100\% - 10\%)\} = (100\% - 100\%) + 15\%$!!"

Strictly speaking, there is an ambiguity about such an expression as "the percentage of marks gained was increased by 50%." This may mean the percentage was increased (say) from 35% to 85%; or it may mean (say) from 40% to 60%. Perhaps in the latter case it would be more correct to say the percentage gained was increased by $50(\%)^2$!

The addition, multiplication and division of fractions are carefully explained with illustrating scales. The writer evidently believes in the efficacy of a thorough drilling in these matters. "With a certain class of teachers, overcome with the desire to make everything in education easy, the complex fraction has fallen into disrepute. These teachers strongly advocate its omission . . . this is done, it is claimed, for the benefit of the student. As a matter of fact, it results in his eternal injury. . . It is pitiable to see average students in Analytical Geometry or Calculus struggling with Arithmetical operations—when they need all their energy to develop the principles of those subjects and should not be forced to dissipate any of it"—in the difficulties of calculation (p. 54).

We recognize some old favourites in Achilles and the Tortoise (p. 166) and the Hundred Cats Killing the Hundred Rats (p. 162). Some of the questions would seem strange to English readers: for instance (p. 416) "A gentleman proposed to plant a vineyard of 10*A*. If he places the vines 6 feet apart; how many more can he plant by setting them in the quincunx order than in the square order, allowing the plot to lie in the form of a square, and no vine to be set nearer its edge than 1 ft. in either case? Answer 1870."

From p. 233, where it is considered in what scale 1331 is denoted by 1000, we go right away in fifty pages to the rectification of the cycloid and quadrature of the spheroids—even triple and elliptic integrals being noted in the text. In many cases the integrations are given in the form of series available for small values of the variable. Facts and figures are given for the conchoid, limaçon, spirals, etc., and generally a great amount of useful work is crowded into the section of mensuration that in England we consider belongs to the Calculus. The length of the solutions is a little wearisome, and it would have been better to have used first approximations only in illustrating methods and principles.

The section on Geometry is very slight and inadequate. Believing (p. vi.) that "one solution, thoroughly analyzed and criticised by a class is worth more than a dozen seen through a cloud of obscurities," the Author gives one notable example of modern descriptive geometry:—If a rectangular hyperbola pass through the middle points of the sides of the triangle ABC and intersect those sides again in P, Q, R respectively, then AP, BQ, CR will co-intersect on the circum-circle. This is indeed an admirable and suggestive theorem. With its study we may well wrestle till break of day. Many interesting issues, such as one-to-one correspondence, are raised. Carnot's theorem teaches us that if a conic intersect the sides of ABC in (P_1, P_2), (Q_1, Q_2), (R_1, R_2) and AP_1, BQ_1, CR_1 are collinear then so also are AP_2, BQ_2, CR_2 (a simple illustration is the Nine Point Circle). Such a conic has for its trilinear equation $\Sigma\alpha/\alpha_1\alpha_2 - \Sigma\beta\gamma/(\beta_1\gamma_2+\gamma_1\beta_2) = O$ (which is analagous to the Cartesian equation to the conic through four points). Make $a\alpha_1 = b\beta_1 = c\gamma_1$, introduce the condition for a rectangular hyperbola, and we easily obtain $\Sigma a/\alpha_2 = O$.

At the end of the volume there are a dozen short biographies of living and dead worthies, those of Sylvester and Cayley being specially interesting. We recommend the Editor of the *Gazette* to consider whether he cannot, after some specially commendable article or solution, introduce a short up-to-date biography of the contributor.

On the whole the book is readable and instructive. The writer has keen enthusiasm for his science, "the practical applications of which have in all ages redounded to the highest happiness of the human race, and by its achievements has bound all the nations of the earth in one common brotherhood of man." Lest, however, any one should boast of being able to read far in nature's infinite book of secrecy let him lay to heart Newton's memorable words (quoted in the biography on p. 501), "I seem to have been only like a boy playing on the sea-shore and diverting myself in now and then finding a smoother pebble or prettier shell than ordinary; whilst the great ocean of truth lay all undiscovered before me."

R. F. DAVIS.

Lehrbuch der Theta-funktionen. By Dr. A. KRAZER. Teubner's "Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften," Band xii. Pp. xxiv. + 509. 24 m.

Dr. Krazer has succeeded in the difficult task of giving a clear deductive account of the complicated formal theory of multiple theta-functions within the compass of a moderate sized text-book. A sufficiently complete account is given of various allied and introductory theories to enable the reader to follow the main argument without much subsidiary study, while numerous references giving the historical development of the subject afford the means of elaborating particular details at pleasure.

The work well fulfils the requirements of a handbook and is distinguished by clearness of style and general elegance of form, though in the latter respect considerable improvement might be introduced by the use of matrix notation. The details of this notation are so easy to grasp that the common reluctance to employ it is remarkable. The singly infinite series $\Sigma \exp.(am^2+2mu)$ (the summation extending over all integral values of m) occurs for the first time in Fourier's

Théorie de la Chaleur, and its introduction as $\theta(u)$ into function-theory was one of the greatest advances made by Jacobi. The chief properties of this function are its uniformity, entirety, and certain periodic properties which, conversely, can serve to define it, and determine its usefulness in the construction of periodic functions. The general theta-function of p arguments may be defined by precisely the same expression where now a is a p-rowed square matrix and m, u are row-letters. To interpret, signs of summation and suffixes must be supplied, pairs of consecutive suffixes being the same; thus am^2, which should more correctly be written mam, means the sum of terms $m_i a_{ij} m_j$. The great advantage of this notation lies not merely in its brevity but in the fact that the letters may be manipulated much like ordinary algebraical symbols, and the periodic properties proved for the single theta-function may be interpreted for the case of p arguments.

The first chapter develops this theory, introducing the "characteristic" and functions of higher order. The second chapter opens with the general theory of the linear transformation of the integers of summation in a multiply infinite series; the transformed series requires the number of solutions of a set of p linear congruences in a simple form involving the invariant-factors of the determinant of coefficients. Application is made of this to transform linearly the arguments and periods of the general theta-function and to obtain many general formulae.

After another modification of the theta-series by means of Fourier's expansion, follows one of the most interesting chapters in the book, showing how the general periodic function may be represented by the quotient of two theta-functions. The line of argument is based on the algebraic relation between $p+1$ functions having $2p$ systems of periods, and the relations among the periods deducible from this. The remaining two chapters of the first part of the book are devoted to Transformation and Complex Multiplication; the latter contains a convenient summary of the chief theorems concerning bilinear forms.

The second part deals mainly with half-integer characteristics (the coefficients in a set of half-periods added to the arguments) and the relations connecting different theta-functions. The particular cases of $p=1$ and $p=2$ are necessarily somewhat curtailed, but what is given is interesting and the list of references with brief descriptions is very comprehensive. In the case $p=2$ the various notations for the 16 theta-functions adopted by most writers are unfortunate in that they do not show clearly from what general formula particular relations are deduced, or, conversely, how to employ a formula without reference to a table. Dr. Krazer either writes the characteristic in full, which is inconvenient, or uses six different symbols for the odd characteristics and their combinations by threes for the even ones, thereby laying undue stress on a distinction which is not essential to the formal theory. If, however, a double suffix notation is used, which is suggested by the matrix notation for a half-period and divides the characteristic horizontally, instead of vertically as in Rosenhain's and Humbert's algorithms, it will be found that many formulae can be written in a general symbolic form from which particular instances can immediately be deduced.

The third part deals with Riemann's special theta-functions of one complex variable in which the p arguments have been replaced by the p finite Abelian integrals for an algebraic curve of deficiency p. The treatment of this portion suffers in respect to clearness and completeness by comparison with the great English treatise on the subject, but lack of space is responsible for many omissions.

It is scarcely necessary to mention that in a volume of this magnificent "Sammlung" every care has been bestowed upon the details of printing so as to render the book as attractive as possible.

R. W. H. T. HUDSON.

Mechanics, Theoretical Applied and Experimental. By W. W. F. PULLEN, Wh. Sc., M.I.M.E., A.M.I.C.E. (Longmans, Green & Co.)

It is an unfortunate fact that most existing textbooks on elementary mechanics are written either, to borrow Professor Perry's classification, exclusively from the 'academic' or exclusively from the 'practical' standpoint; and are the poorer for this limitation. The best academic textbooks are admirably precise and clear, but unfortunately disregard entirely the historical and practical aspects of the subject. The baser kind of academic cram-book is equally silent as to matters of practical utility or historical interest, and is often disgracefully unsound on matters of fundamental principle, giving for instance a 'fudge' of Atwood's

machine by $$\text{mass moved} = m + m',$$ $$\text{moving force} = (m - m')g.$$

On the other hand, the practical textbooks, though they frequently draw attention to applications of great interest (and in this respect the book under review deserves high praise) are too often weak on the theoretical side, partly no doubt because they encounter a difficulty from which the purely 'academic' writer escapes.

A theory, embodied in a theorem, states that, given certain premises, a certain conclusion follows. This is all that the 'academic' writer is concerned with. For instance, if a weightless, flexible string passes round a smooth peg, the tensions on either side are the same. Now the constant repetition of these qualifying premises is tiresome, and there is a temptation to say "the tension of a string on either side of a pulley is the same." The original theorem is obviously inapplicable in practice; the modified form is free from this objection, but is untrue.

A writer, whose aim is practice, is tempted, adapting the medieval view that a doctrine may be theologically false, but philosophically true, to say that the proposition "the tension of a string is the same on either side of a pulley" is theoretically true, but practically false. The enunciation of conclusions which are based upon certain definite premises or conditions, as if they were absolutely and unconditionally true, is responsible for much of the contempt and dislike which the "practical man," to his own hindrance, is apt to show for theory.

Mr. Pullen states at page 51 and again (in block type) at page 68: "The mutual pressure between two surfaces in contact is perpendicular to the surfaces at the points of contact"; and at page 161: "The velocity of a body falling down any curved path is the same whatever the shape of the path."

The writer of a textbook should have the same aim as a conveyancer; each of his sentences should be capable of only one possible meaning. This should be so not merely for the sake of the subject, but for the sake of the English language. I should like to quote a few illustrative gems which are **not** from Mr. Pullen's book, but from other "practical works."

(i) "Velocity is said to be uniform when it moves over equal distances in the same time, and it is variable when it is continually altering."

(ii) "The radius of curvature in a beam of uniform strength and depth is a circle."

(iii) "Since generally force = mass × velocity (when force is measured by momentum)."

In Mr. Pullen's book, though it is generally carefully written, we come across a loose statement here and there. For example, the symbol g is used in three distinct senses without warning. On page 206 we find, "The student should try to remember that a force of g dynes is equivalent to a force of 1 gramme, and a force of g poundals is equivalent to a force of 1 pound." Could one severely blame a student who remembered that 32 dynes = 1 gramme? At page 157 and elsewhere we find "Force × g = mass moved × acceleration produced," a statement which, on the most favourable construction, is dangerous; for g, which heretofore denoted an acceleration, has suddenly become a mere number; and the transition is unexplained. There is some confusion between mass and weight at page 157, and on page 355 this leads to a positive mistake, for there we find, in block type, "Density = mass of unit of volume. Pressure of a fluid on a surface immersed in it = area of surface × depth of centre of gravity of surface × density of fluid."

In teaching the elements of mechanics to beginners, the maxim *Entia non sunt multiplicanda* should be translated: "Postpone the use of the word mass as long as you can."

Again, while fully admitting that a definition should generally be the coping stone rather than the foundation of a structure, I cannot consider the statement (p. 190), "The Radius of Gyration of a body is the distance from the axis of rotation at which the mass of a body can be supposed to be concentrated to produce the same result" as well fitted for either purpose. On page 42 we have a "general method for finding the resultant of any number of forces in one plane" without any warning of the difficulty which will arise if the system of forces happens to be equivalent to a couple. Such instances as these induce a feeling of lack of confidence; and even in exclusively practical matters Mr. Pullen is not always an absolutely reliable guide. On page 94 we read: "If the centre of gravity of

a balance is above the point of support . . . it would be altogether useless." But the so-called 'accelerating' balance is a very common object in shops. At page 23 the absence of an 'elastic' or 'plastic' stage in string is inferred from an experiment which appears rather to indicate the gradual pulling out of the 'twist.' It would be interesting to repeat the experiment with horsehair, unspun thread, and a stiff helical spring.

I have made the preceding criticisms because they relate to matters which can be put right by a little revision, and I now have the more pleasant task of pointing out the merits of the book. Much care has evidently been expended in selecting illustrations from real life of the mechanical principles discussed. To mention only three instances, that ingenious as well as useful appliance Weston's differential pulley is most thoroughly discussed with very numerous examples: the 'platform' weighing machine is well and clearly described, and directions are given for the repetition by Galileo's own method of his experiments on acceleration. Mr. Pullen lays much stress on actual experiment. There can hardly be two opinions as to the absolute justice of this view. Nothing is more disheartening than to find, as constantly happens, that boys have 'done mechanics,' and can accordingly sometimes work out a mechanical problem and sometimes obtain a correct *numerical* result, but give it a wrong description. When the reaction at a support is stated to be 400 foot pounds, or we are told * that "a cubic foot of water moving at 113 feet per second has momentum equal to 798,062 foot lbs.," we have a demonstration that in the pupil's view, force, momentum, energy, acceleration, and so on, are only words, and obviously, in the republic of letters, one word is as good as another.

Though, for the reasons I have stated, the book is not one to put unreservedly into the hands of a beginner until it has undergone revision; yet no teacher can take it up without deriving much benefit from the numerous valuable hints and examples which it contains. The book is remarkably well printed with clear and excellent diagrams.

C. S. JACKSON.

Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung Statistik und Lebensversicherung. Vol. II. By EMANUEL CZUBER. 1903. (Teubner.)

"The calculation of probabilities and their application to the adjustment of error, Statistics and Life Insurance," is a well-printed book, setting forth clearly and concisely the principles underlying the science of Life Insurance. Commencing with an exposition of the mathematical probabilities of death and survivorship, the author goes on to explain the functions known as the force of mortality and the central death rate, and the method of calculating the expectation of life. Then follow some interesting methods of representing geometrically the effects of mortality upon masses of individuals, methods differing from those employed by English actuaries.

The systems of constructing and graduating mortality tables are set forth at length. The graduation formulas of Gompertz, Makeham, Wittstein, Woolhouse, and Karup are analysed, and the graphic method advocated by Dr. Sprague is also explained.

The chapter on Invalidity and Mortality is interesting, as it deals with a subject of special importance in Germany, and not unlikely in future to call for consideration in this country. The analysis of the formulas for calculating annuity values is clear and complete, but we think there is a slight typographical error on page 458, where $m\left(\frac{1}{v^{\frac{1}{m}}}-1\right)^{\frac{1}{m}}$ is put equal to $m\{(1+i)^{\frac{1}{m}}-1\}$. Formulas are also given for annuities payable during "Activity" and "Invalidity." It is noteworthy in this connection that the commutation column N_x is based on the relation $N_x = D_x + D_{x+1} + D_{x+2} + \text{etc.}$ instead of $N_x = D_{x+1} + D_{x+2} + \text{etc.}$, as used in this country, and this also in the case of the Hm. Mortality Table of the 20 British offices in which the text-book graduation is adopted. Seeing that the column of values of μx from this table is also given, we are somewhat surprised that no mention is made of the very valuable formulas of approximate

* This statement got into print, but is a good example of the sort of thing I refer to.

summation applied to actuarial purposes by Mr. G. F. Hardy and introduced by Mr. King into Part II. of the text-book of the Institute of Actuaries.

Zillmer's method of dealing with the problem of providing for the special expenditure incurred in the first year of insurance—a problem to which Dr. Sprague among English actuaries has devoted especial attention—is given, the basis being as follows:—"The insured pays annually an equal gross premium whereas the net premium is constant from the second year, but calculated as less for the first year than the following years by an amount a. This amount which enters into the additional payment, and may therefore serve for other than pure insurance purposes, is intended to cover the preliminary expenses. The net premium for the first year, however, must, if the first principles of variable premium payments be satisfied, be at least equal to the natural premium. Obviously the reduction of the first premium involves a corresponding increase of the following premiums. The notion underlying this method of calculation may be amplified in that the first addition may be spread over several years instead of over one. Dealing, for example, with a whole life insurance with a premium payable annually throughout life; from the second year the premium amounts to P_x, then in the first year it is $P_x - a$, and the present value of all the premiums is expressed by $P_x a_x - a$. Equating this to the value A_x of the insurance we have $P_x = \frac{A_x + a}{a_x}$, while the premium remaining constant throughout the whole term is $P'_x = \frac{A_x}{a_x}$.

The first essential requirement is that a must be taken less than P_x. This leads to the condition $a < \frac{A_x + a}{a_x}$.

$$a\left(1 - \frac{1}{a_x}\right) < P'_x,$$

$$a < \frac{a_x}{a_x - 1} P'_x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The second requirement formulated above is $P_x - a \geqq \frac{C_x}{D_x}$.

$$P'_x + \frac{a}{a_x} - a \geqq \frac{C_x}{D_x},$$

$$a \leqq \frac{a_x}{a_x - 1}\left(P'_x - \frac{C_x}{D_x}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If (1) is fulfilled (2) is fulfilled also; hence the right hand member of (2) becomes an upper limit for the amount a, a limit which varies (increasingly) with the age. It is another question whether a selected with this limit is always sufficient to cover the preliminary expenses or whether this is at least to be obtained with the aggregate of the insurances."

It must be borne in mind that in the notation employed throughout the book $a_x = \frac{D_x + D_{x+1} + \text{etc.}}{D_x}$, or $1 + a_x$ in the English notation, it is the value of an annuity due; hence in the English notation formula (2) becomes

$$a \leqq \frac{1 + a_x}{a_x}\left(P'_x - \frac{C_x}{D_x}\right).$$

From the Hm. Table the upper limit of a when $x = 30$ is given as $a \leqq 0{\cdot}01033$, about 1 % of the capital sum assured. When $x = 50$, $a \leqq 0{\cdot}02321$, about 2·3 % of the capital sum assured.

Next follow chapters on insurances with guaranteed return of premiums paid and on policy values, the latter containing a brief explanation of the formulas applicable to a change in the form of an insurance; for example, from a whole life to an endowment assurance. The final chapter deals with the mean risk in life insurance and the solution of the general problem of estimating the risk incurred by an insurance office as regards its insured at a given point of time. The problem is dealt with at some length, and is likely to be of interest to English actuaries.

Appended to the book are the following tables:—The value of the function $\phi(\gamma)=\frac{2}{\sqrt{\pi}}\int_0^{\gamma}e^{-v^2}dt$; the German mortality table for male and female lives; the Hm. Table 20 offices 3½ %; Tables M. and W. I. of 23 German offices 3½ %; and invalidity and mortality tables of German railway officials 3½ %.

The volume is deserving of careful study, and is ably compiled. It is not entirely free from printer's errors, but these are commendably few, having regard to the difficult nature of the subject and the necessarily complicated nature of the symbols employed.

W. R. STRONG, F.I.A.,
London Guarantee and Accident Co.

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Zeitschrift für Geschichte der Mathematischen Wissenschaften. Sonderabdruck aus Bibliotheca Mathematica. III. Folge IV. 1903. *Peter Guthrie Tait; His Life and Works.* By A. MACFARLANE. pp. 185-200. V.

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